

Radically solvable graphs

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Abstract

A 2-dimensional framework is a straight line realisation of a graph in the Euclidean plane. It is radically solvable if the set of vertex coordinates is contained in a radical extension of the field of rationals extended by the squared edge lengths. We show that the radical solvability of a generic framework depends only on its underlying graph and characterise which planar graphs give rise to radically solvable generic frameworks. We conjecture that our characterisation extends to all graphs.

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1 Introduction

Many systems of polynomial equations which are of practical interest can be represented by a graph. An important example occurs in computer aided design (CAD) when the location of the geometric elements in a drawing such as points and lines (corresponding to vertices in the graph) are determined by relationships between them such as tangency, coincidence and the relative separations or angles between them (corresponding to edges in the graph). The ability to solve such systems of equations rapidly allows a design engineer to modify input parameters such as the values for the separations or angles

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(collectively called "dimensions" in a dimensioned drawing) and to realise a computer model for many variants of a basic design [9]. Most modern CAD systems incorporate the ability to solve these so-called dimensional constraint equations, see for example [11].

A simple example of dimensional constraint equations is provided by points in a plane with certain specified relative distances. Both the equations and a particular solution can be represented by a framework (G, p) where G is a graph and p is a vector comprising of all the coordinates of the points. The graph G has a vertex for each point and an edge for each specified distance. Since the coordinates of the points are specified in (G, p) it is a simple matter to determine the relative distance corresponding to any edge of G . The framework (G, p) therefore represents both a system of polynomial equations and a particular solution to these equations. We will call these equations the framework equations - they correspond to the dimensional constraint equations referred to above. In general the framework vector p will be just one of the many possible solutions to the framework equations. (Estimates on the number of solutions have been obtained by several authors, see for example [1, 5, 12].)

Efficient algorithms for solving the framework equations are extremely useful. A particularly desirable case is when there are only a finite number of solutions, and these solutions can be expressed as a sequence of square, or higher power, roots of combinations of the squared edge distances. Such frameworks are said to be quadratically solvable (or ruler-and-compass-constructible [3]) and radically solvable, respectively. We will consider the problem of determining which generic frameworks are quadratically or radically solvable.

The condition that the framework equations should have only finitely many solutions is equivalent to the statement that the framework is rigid. This property has been extensively studied and we refer the reader to [15] for an excellent survey of the area. Previous work on quadratic/radical solvability [9, 10] considered generic frameworks which are minimally rigid i.e. cease to be rigid when any edge is removed. A conjectured characterisation of quadratically/radically solvable minimally rigid generic frameworks was given in [9] and this conjecture was verified for the special case when the underlying graph is 3-connected and planar in [10].

We will extend the study of quadratic and radical solvability to include generic frameworks which are rigid but not necessarily minimally rigid. We first show in Lemma 5.2 that the quadratic or radical solvability of a generic

framework depends only on the underlying graph. This means that if a graph is quadratically or radically solvable then there will be a quadratic or radical solution to the corresponding system of framework equations for any sufficiently general but consistent set of input distances. We next consider globally rigid graphs i.e. graphs for which every generic realisation is a unique solution to the corresponding framework equations. We show in Theorem 6.2 that all such graphs are quadratically solvable.

We develop a reduction scheme in Section 7 which shows how the radical or quadratic solvability of a rigid graph is related to the corresponding property for a derived graph which may be chosen to be minimally rigid. We use this and the main result of [10] to show in Theorem 7.5 that a rigid 3-connected planar graph is radically solvable if and only if it is globally rigid. This leads us to consider rigid graphs which are not 3-connected i.e. graphs $G = (V, E)$ which can be separated into two subgraphs $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{u, v\}$. We show in Theorem 8.1 that the radical or quadratic solvability of G is determined by the corresponding property of $G_1 + uv$ and $G_2 + uv$ when G_1 and G_2 are both rigid, and of $G_1 + uv$ and G_2 when G_1 is not rigid and G_2 is minimally rigid. We use this analysis to give a constructive definition for a family of quadratically solvable graphs \mathcal{F} . We conjecture that every radically solvable graph belongs to \mathcal{F} and prove in Theorem 9.3 that this holds for planar graphs.

2 Definitions and Notation

All graphs considered are finite and without loops or multiple edges. Given a graph $G = (V, E)$ and two vertices $u, v \in V$ we use $G + uv$ to denote the graph $(V, E \cup \{uv\})$. A *complex (real) realisation* of G is a map p from V to \mathbb{C}^2 (\mathbb{R}^2). We also refer to the ordered pair (G, p) as a complex (real) *framework*. Although we are mainly concerned with real frameworks, we will work with complex frameworks since most of our methods require an algebraically closed field and our results can still be applied to the special case of real frameworks. Henceforth we assume that all frameworks not specifically described as real, are complex. A framework (G, p) is *generic* if the set of all coordinates of the points $p(v)$, $v \in V$, is algebraically independent over \mathbb{Q} .

Let $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$. Given a realisation (G, p) of G in \mathbb{C}^2 and two vertices $v_i, v_j \in V$ with $p(v_i) - p(v_j) = (a, b)$ put $d_p(v_i, v_j) = a^2 + b^2$ and $d_p(e) = d_p(v_i, v_j)$ when $e = v_i v_j \in E$. Two

realisations (G, p) and (G, q) are *equivalent* if $d_p(e) = d_q(e)$ for all $e \in E$, and are *congruent* if $d_p(v_i, v_j) = d_q(v_i, v_j)$ for all $v_i, v_j \in V$. The *rigidity map* $d_G : \mathbb{C}^{2n} \rightarrow \mathbb{C}^m$ is defined by putting $d_G(p) = (d_p(e_1), d_p(e_2), \dots, d_p(e_m))$. Thus (G, p) and (G, q) are equivalent if and only if $d_G(p) = d_G(q)$. Note that, if (G, p) and (G, q) are real frameworks, then they are equivalent if and only if they have the same edge lengths and they are congruent if and only if we can transform one to the other by applying an isometry of \mathbb{R}^2 i.e. a translation, rotation or reflection of the Euclidean plane.

A framework is *globally rigid* if all equivalent frameworks are congruent to it. A real framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that every real framework (G, q) which is equivalent to (G, p) and satisfies $d(p(v) - q(v)) = \|p(v) - q(v)\|^2 < \epsilon$ for all $v \in V$, is congruent to (G, p) .¹ It is known that both the rigidity and the global rigidity of a generic framework depend only on its underlying graph. We say that a graph G is *rigid* if some, or equivalently every, generic real realisation of G is rigid, and that G is *globally rigid* if some, or equivalently every, generic realisation of G is globally rigid.

Let K, L be fields with $K \subseteq L$. Then L is a *radical extension* of K if there exist fields $K = K_1 \subset K_2 \subset \dots \subset K_t = L$ such that for all $1 \leq i < t$, $K_{i+1} = K_i(x_i)$ with $x_i^{n_i} \in K_i$ for some natural number n_i . The field L is a *quadratic extension* of K if it is a radical extension with $n_i = 2$ for all $1 \leq i < t$. We say that $L : K$ is *radically solvable*, respectively *quadratically solvable*, if L is contained in a radical, respectively quadratic, extension of K . A realisation (G, p) of a rigid graph G is *radically solvable*, respectively *quadratically solvable*, if there exists a congruent realisation (G, q) such that $\mathbb{Q}(q) : \mathbb{Q}(d_G(q))$ is radically, respectively quadratically, solvable.

3 Field extensions and algebraic varieties

The above definitions of radically and quadratically solvable field extensions immediately imply the following result.

Lemma 3.1 *Let $K \subseteq L \subseteq M$ be fields. Then $M : K$ is radically, respectively quadratically, solvable if and only if $M : L$ and $L : K$ are both radically, respectively quadratically, solvable.*

¹Equivalently, a real framework (G, p) is rigid if every continuous motion of the points $p(v)$, $v \in V$, in \mathbb{R}^2 which preserves the edge distances results in a framework which is congruent to (G, p) .

We next recall some definitions and results from Galois theory. We adopt the notation of [13] and refer the reader to this text for further information on the subject.

Given a field extension $L : K$ we use $[L : K]$ to denote the *degree* of the extension i.e. the dimension of L as a vector space over K . The extension is *finite* if it has finite degree. It is *normal* if L is the splitting field of some polynomial over K . When $L : K$ is finite, a *normal closure of L over K* is a field N such that $L \subseteq N$, $N : K$ is normal, and, subject to these conditions, N is minimal with respect to inclusion. It is known that normal closures exist, are finite, and are unique up to isomorphism, see [13, Theorem 11.6]. The *Galois group* $\Gamma(L : K)$ is the group of all automorphisms of L which leave K fixed. Galois theory gives us the following close relationship between radically/quadratically solvable extensions and Galois groups, see [13, Theorems 15.3, 18.18].²

Theorem 3.2 *Let K be a field of characteristic zero and $N : K$ be a normal field extension. Then*

- (a) *$N : K$ is radically solvable if and only if $\Gamma(N : K)$ is a solvable group.*
- (b) *$N : K$ is quadratically solvable if and only if $\Gamma(N : K)$ is a 2-group.*

Our next result allows us to decide whether a field extension $L : K$ is radically, respectively quadratically, solvable by applying Theorem 3.2 to its normal closure. This will be used to show that the radical or quadratic solvability of a generic framework depends only on its underlying graph.

Lemma 3.3 *Let K be a field of characteristic zero and $L : K$ be a finite field extension. Let N be a normal closure of L over K . Then $L : K$ is radically, respectively quadratically, solvable if and only if $N : K$ is radically, respectively quadratically, solvable.*

Proof. Sufficiency follows from Lemma 3.1. To prove necessity we assume that $L : K$ is radically, respectively quadratically, solvable. Then L is contained in a radical, respectively quadratic, extension M of K . Let P be a normal closure of M over K , see Figure 1(a). Since $L \subseteq M$ and normal closures are unique up to isomorphism, we may suppose that $N \subseteq P$. Since M is a radical, respectively quadratic, extension of K and P is a normal closure

²The references to [13] in this section only give results on radically solvable extensions, but similar proofs work for the special case of quadratically solvable extensions.

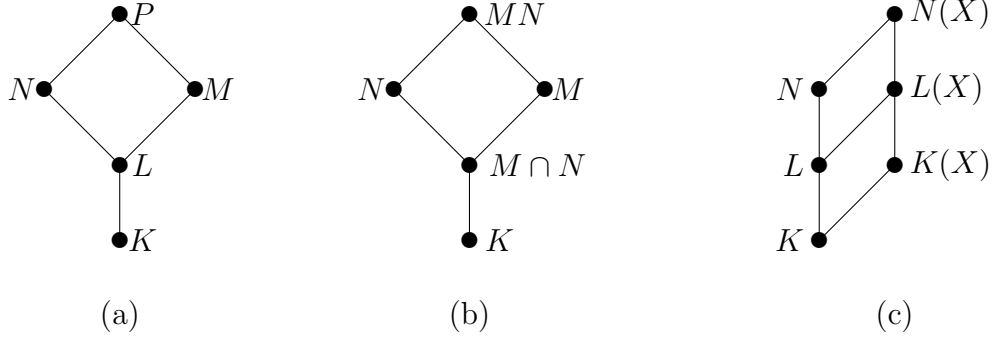


Figure 1: The field extensions of Lemmas 3.3, 3.4 and 3.5

of M over K , [13, Lemma 15.4] implies that P is also a radical, respectively quadratic, extension of K . Since $N \subseteq P$, $N : K$ is radically, respectively quadratically, solvable. •

Suppose M, N are field extensions of a field K which are both contained in a common extension P of K . Then MN denotes the smallest subfield of P which contains both M and N . We will need the following result from Galois Theory, see for example [8, Proposition 3.18].

Lemma 3.4 *Let K be a field of characteristic zero and M, N be field extensions of K which are both contained in a common extension of K . Suppose that N is a normal extension of K . Then $MN : M$ and $N : M \cap N$ are normal extensions, and $\Gamma(MN : M)$ and $\Gamma(N : M \cap N)$ are isomorphic groups.*

Given a field K we use $K[X_1, X_2, \dots, X_n]$ to denote the ring of polynomials in the indeterminates X_1, X_2, \dots, X_n with coefficients in K and $K(X_1, X_2, \dots, X_n)$ to denote its field of fractions.

Lemma 3.5 *Let $L : K$ be a finite field extension with $\mathbb{Q} \subseteq K \subseteq L \subset \mathbb{C}$, and N be the normal closure of $L : K$ in \mathbb{C} . Let $X = (X_1, X_2, \dots, X_n)$ be a vector of indeterminates. Then $N(X)$ is a normal closure of $L(X)$ over $K(X)$ and $\Gamma(N : K)$ is isomorphic to $\Gamma(N(X) : K(X))$. Furthermore, $L : K$ is radically, respectively quadratically, solvable if and only if $L(X) : K(X)$ is radically, respectively quadratically, solvable.*

Proof. Let a_1, a_2, \dots, a_m be a basis for $L : K$, f_i be the minimum polynomial of a_i over K , R_i be the set of all complex roots of f_i , and $R = \bigcup_{i=1}^m R_i$. Then

$N = L(R)$. Since X_1, X_2, \dots, X_n are indeterminates, a_1, a_2, \dots, a_m is also a basis for $L(X) : K(X)$ and f_i is the minimum polynomial of a_i over $K(X)$. Thus $L(R)(X) = N(X)$ is a normal closure of $L(X) : K(X)$. We now apply Lemma 3.4 with $M = K(X)$. We have $NK(X) = N(X)$ and $N \cap K(X) = K$. Hence $\Gamma(N : K)$ is isomorphic to $\Gamma(N(X) : K(X))$.

The final part of the lemma now follows from Theorem 3.2 and Lemmas 3.3 and 3.4. •

Our next result is an application of the previous lemmas. We will use it to determine whether generic realisations of graphs with small separating sets of vertices are radically or quadratically solvable.

Lemma 3.6 *Suppose that $X = (X_1, X_2, \dots, X_r)$, $Y = (Y_1, Y_2, \dots, Y_s)$ and $Z = (Z_1, Z_2, \dots, Z_t)$ are vectors of indeterminates, $f = (f_1, f_2, \dots, f_m) \in \mathbb{Q}[X, Y]^m$ and $g = (g_1, g_2, \dots, g_n) \in \mathbb{Q}[Y, Z]^n$, and $\mathbb{Q}(X, Y, Z)$ is a finite extension of $\mathbb{Q}(f, g)$. Then $\mathbb{Q}(X, Y, Z) : \mathbb{Q}(f, g)$ is radically, respectively quadratically, solvable if and only if $\mathbb{Q}(f, Y, Z) : \mathbb{Q}(f, g)$ and $\mathbb{Q}(X, Y) : \mathbb{Q}(f, Y)$ are both radically, respectively quadratically, solvable.*

Proof. This follows from Lemma 3.1 (which tells us that $\mathbb{Q}(X, Y, Z) : \mathbb{Q}(f, g)$ is radically, respectively quadratically, solvable if and only if $\mathbb{Q}(f, Y, Z) : \mathbb{Q}(f, g)$ and $\mathbb{Q}(X, Y, Z) : \mathbb{Q}(f, Y, Z)$ are both radically, respectively quadratically, solvable) and Lemma 3.5 (which tells us that $\mathbb{Q}(X, Y, Z) : \mathbb{Q}(f, Y, Z)$ is radically, respectively quadratically, solvable if and only if $\mathbb{Q}(X, Y) : \mathbb{Q}(f, Y)$ is radically, respectively quadratically, solvable). •

Our final result of this section concerns algebraic varieties. We will use it to show, amongst other things, that globally rigid graphs are quadratically solvable.

Lemma 3.7 *Let K be a field with $\mathbb{Q} \subseteq K \subset \mathbb{C}$, and let $S = \{f_1, \dots, f_m\} \subset K[X]$, where $X = (X_1, \dots, X_n)$ is a vector of indeterminates. Let $I \subset K[X]$ be the ideal generated by S and $W = \{x \in \mathbb{C}^n : f_i(x) = 0 \text{ for all } 1 \leq i \leq m\}$. Let $I_1 = I \cap K[X_1]$. Then I_1 is an ideal of $K[X_1]$ and is generated by a single polynomial $h_1 \in K[X_1]$. Furthermore, if W is non-empty and finite and $h_1(a) = 0$ for some $a \in \mathbb{C}$, then there exists an $x = (x_1, x_2, \dots, x_n) \in W$ such that $x_1 = a$.*

Proof. It is easy to see that I_1 is an ideal of $K[X_1]$. It is generated by a single polynomial since $K[X_1]$ is a principal ideal domain. The final part of the lemma follows from the work of Kalkbrener [6]. We include an outline of his proof for completeness. Let $I_s = I \cap K[X_1, \dots, X_s]$ for all $1 \leq s \leq n$ and $W_s = \{x \in \mathbb{C}^s : f(x) = 0 \text{ for all } f \in I_s\}$. It will suffice to show that for all $2 \leq s \leq n$ and all $b \in W_{s-1}$, there exists a $c \in \mathbb{C}$ such that $(b, c) \in W_s$. Let $I_s(b) = \{f(b, X_s) : f \in I_s\}$. Then $I_s(b)$ is an ideal of $K(b)[X_s]$. It follows from [6, Theorems 3] that $I_s(b)$ is generated by a non-constant polynomial h_s in $K(b)[X_s]$. We can now choose a $c \in \mathbb{C}$ with $h_s(c) = 0$ to obtain the required element $(b, c) \in W_s$. •

4 Standard positions

Given a generic framework (G, p) it will be useful to identify a particular congruent framework (G, q) with the property that (G, p) is radically, or quadratically, solvable if and only if $\mathbb{Q}(q)$ is contained in a radical, or quadratic, extension of $\mathbb{Q}(d_G(q))$. The following result will enable us to do this.

Lemma 4.1 *Suppose that (G, p) is a generic realisation of a graph $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$ and $n \geq 3$. Then there are exactly four realisations (G, q_j) , $1 \leq j \leq 4$, which are congruent to (G, p) and have $q_j(v_1) = (0, 0)$ and $q_j(v_2) = (0, z)$ for some $z \in \mathbb{C}$. Furthermore, we have $\mathbb{Q}(q_i) = \mathbb{Q}(q_j)$ for all $1 \leq i < j \leq 4$.*

Proof. The assertion that there are exactly four such realisations (G, q_i) is a special case of [5, Corollary 5.3]. The assertion that $\mathbb{Q}(q_i) = \mathbb{Q}(q_j)$ follows from the fact that we can order the q_j such that, if $q_1(v_i) = (x_i, y_i)$ for all $v_i \in V$, then $q_2(v_i) = (-x_i, y_i)$, $q_3(v_i) = (x_i, -y_i)$ and $q_4(v_i) = (-x_i, -y_i)$ for all $v_i \in V$. •

Given a graph G and vertices v_1, v_2 of G , we say that a realisation (G, q) of G is in *standard position with respect* (v_1, v_2) if $q(v_1) = (0, 0)$ and $q(v_2) = (0, z)$ for some $z \in \mathbb{C}$, and is *quasi-generic* if it is congruent to a generic realisation of G .

Lemma 4.2 *Suppose that (G, p) is a quasi-generic realisation of a rigid graph $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$ and $p(v_i) = (x_i, y_i)$ for $1 \leq i \leq n$. Suppose further that (G, p) is in standard position with respect to (v_1, v_2) , i.e. $x_1 = y_1 = x_2 = 0$. Then $\{y_2, x_3, y_3, \dots, y_n\}$ is algebraically independent over \mathbb{Q} .*

Proof. This follows immediately from [5, Lemma 5.4]. •

Lemma 4.3 *Suppose that (G, p) is a generic realisation of a rigid graph $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$ and $n \geq 3$, and (G, q) is a congruent realisation in standard position with respect to (v_1, v_2) . Then (G, p) is radically, respectively quadratically, solvable if and only if $\mathbb{Q}(q) : \mathbb{Q}(d_G(q))$ is radically, respectively quadratically, solvable.*

Proof. Sufficiency follows immediately from the definition of radically, respectively quadratically, solvable frameworks. To prove necessity we suppose that (G, p) is radically, respectively quadratically, solvable. Replacing (G, p) by a congruent framework if necessary, we may assume that $\mathbb{Q}(p)$ is itself contained in a radical, respectively quadratic, extension L of $\mathbb{Q}(d_G(p))$. We can construct a framework (G, q) satisfying the hypotheses of the lemma by putting $\tilde{q}(v_i) = p(v_i) - p(v_1)$ for all $v_i \in V$, and

$$q(v_i) = \begin{pmatrix} y/d_0 & -x/d_0 \\ x/d_0 & y/d_0 \end{pmatrix} \tilde{q}(v_i)$$

for all $v_i \in V(G)$, where $\tilde{q}(v_2) = (x, y)$ and $d_0^2 = x^2 + y^2$. By Lemma 4.1, it will suffice to show that for this q , $\mathbb{Q}(q)$ is contained in a radical, respectively quadratic, extension of $\mathbb{Q}(d_G(q))$. Let $K = \mathbb{Q}(p, d_0)$. The definitions of \tilde{q} and q imply that $\mathbb{Q}(\tilde{q}) \subseteq \mathbb{Q}(p)$ and hence that $\mathbb{Q}(q) \subseteq K$. We have $[K : \mathbb{Q}(p)] \leq 2$ since $d_0^2 = x^2 + y^2$ and $x, y \in \mathbb{Q}(p)$. Hence $L(d_0)$ is a radical, respectively quadratic, extension of $\mathbb{Q}(d_G(p))$ which contains K . Since $\mathbb{Q}(q) \subseteq K$ and $d_G(p) = d_G(q)$, $\mathbb{Q}(q) : \mathbb{Q}(d_G(q))$ is radically, respectively quadratically, solvable. •

5 Quadratically and radically solvable graphs

We first show that a quasi-generic realisation of a rigid graph gives rise to a finite field extension when it is in standard position.

Lemma 5.1 *Suppose that $G = (V, E)$ is a rigid graph and that (G, p) is a quasi-generic realisation of G in standard position with respect to two vertices $v_1, v_2 \in V$. Then $\mathbb{Q}(p) : \mathbb{Q}(d_G(p))$ is a finite field extension.*

Proof. It is easy to see that $\mathbb{Q}(d_G(p)) \subseteq \mathbb{Q}(p)$. By [5, Lemma 5.4], $\mathbb{Q}(p)$ and $\mathbb{Q}(d_G(p))$ have the same algebraic closure. This implies that each coordinate of p is a root of a polynomial with coefficients in $\mathbb{Q}(d_G(p))$ and hence $[\mathbb{Q}(p) : \mathbb{Q}(d_G(p))]$ is finite. \bullet

We next show that radical and quadratic solvability are generic properties of frameworks i.e. they depend only on the underlying graph when the given realisation is generic.

Lemma 5.2 *Suppose (G, p) and (G, p') are generic realisations of a rigid graph $G = (V, E)$. Then (G, p) is radically, respectively quadratically, solvable if and only if (G, p') is radically, respectively quadratically, solvable.*

Proof. Let $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$. Let (G, q) and (G, q') be two frameworks in standard position with respect to (v_1, v_2) which are congruent to (G, p) and (G, p') , respectively. Put $q(v_i) = (x_{2i-1}, x_{2i})$ and $q'(v_i) = (x'_{2i-1}, x'_{2i})$ for $1 \leq i \leq n$. We associate a pair of indeterminates (X_{2i-1}, X_{2i}) with each vertex $v_i \in V$, putting $X_1 = X_2 = X_3 = 0$ to represent a framework in standard position. Let $X = (X_4, X_5, \dots, X_{2n})$ and $D_G(X) = (f_1, f_2, \dots, f_m)$ where $f_i = (X_{2j-1} - X_{2k-1})^2 + (X_{2j} - X_{2k})^2$ when $e_i = v_j v_k$. Since (G, q) and (G, q') are quasi-generic, Lemma 4.2 implies that $\{x_3, x_4, \dots, x_{2n}\}$ and $\{x'_3, x'_4, \dots, x'_{2n}\}$ are both algebraically independent over \mathbb{Q} . Hence $\mathbb{Q}(q) : \mathbb{Q}(d_G(q))$ and $\mathbb{Q}(q') : \mathbb{Q}(d_G(q'))$ are both isomorphic to $\mathbb{Q}(X) : \mathbb{Q}(D_G(X))$.³

Let N_q , $N_{q'}$ and N_X be normal closures of $\mathbb{Q}(q) : \mathbb{Q}(d_G(q))$, $\mathbb{Q}(q') : \mathbb{Q}(d_G(q'))$ and $\mathbb{Q}(X) : \mathbb{Q}(D_G(X))$, respectively. Then $N_q : \mathbb{Q}(d_G(q))$ and $N_{q'} : \mathbb{Q}(d_G(q'))$ are both isomorphic to $N_X : \mathbb{Q}(D_G(X))$ and hence are isomorphic to each other. It follows that $\Gamma(N_q : \mathbb{Q}(d_G(q)))$ and $\Gamma(N_{q'} : \mathbb{Q}(d_G(q')))$ are isomorphic groups. The lemma now follows by applying Theorem 3.2 and Lemma 3.3. \bullet

³Two field extensions $L : K$ and $L' : K'$ are isomorphic if there exists a field isomorphism from L to L' which maps K onto K' .

This result allows us to define a rigid graph to be *radically*, respectively *quadratically*, *solvable* if some (or equivalently every) generic realisation of G is radically, respectively quadratically, solvable. Lemmas 4.3 and 5.2 imply that this definition agrees with the one given for the radical and quadratic solvability of minimally rigid graphs in [10, Definition 3.1].

6 Globally rigid graphs

Two vertices v_i, v_j of a rigid graph G are *globally linked* if for each generic realisation (G, p) and every equivalent realisation (G, q) we have $d_p(v_i, v_j) = d_q(v_i, v_j)$.

Lemma 6.1 *Let (G, p) be a quasi-generic realisation of a rigid graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$. Suppose that $v_a, v_b \in V$ are globally linked in G . Then $d_p(v_a, v_b) \in \mathbb{Q}(d_G(p))$.*

Proof. We may suppose that (G, p) is in standard position with respect to v_1, v_2 . Let $K = \mathbb{Q}(d_G(p))$. We again associate a pair of indeterminates (X_{2i-1}, X_{2i}) with each vertex $v_i \in V$, putting $X_1 = X_2 = X_3 = 0$ to represent a framework in standard position. Let $f_i = (X_{2j-1} - X_{2k-1})^2 + (X_{2j} - X_{2k})^2 - d((p(v_j) - p(v_k)))$ for each $e_i = v_j v_k \in E$. We introduce a new indeterminate X_{2n+1} which represents the ‘distance’ between v_a and v_b and put $f_{m+1} = X_{2n+1} - (X_{2a-1} - X_{2b-1})^2 - (X_{2a} - X_{2b})^2$. Let $X = (X_4, X_5, \dots, X_{2n+1})$. Let I be the ideal of $K[X]$ generated by the polynomials f_1, f_2, \dots, f_{m+1} and let $I_{2n+1} = I \cap K[X_{2n+1}]$. Then I_{2n+1} is generated by a single polynomial $h_{2n+1} \in K[X_{2n+1}]$, and every zero of h_{2n+1} in \mathbb{C} extends to a zero of I in \mathbb{C}^{2n+1} by Lemma 3.7. Since v_a, v_b are globally linked in G , $d_p(v_a, v_b)$ must be the unique zero of h_{2n+1} . Thus $h_{2n+1} = (X_{2n+1} - d_p(v_a, v_b))^t$ for some positive integer t . Since $h_{2n+1} \in K[X_{2n+1}]$ this implies that $d_p(v_a, v_b) \in K$.

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Theorem 6.2 *Every globally rigid graph is quadratically solvable.*

Proof. Let $G = (V, E)$ be a globally rigid graph with $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$. Let (G, p) be quasi-generic realisation of G which is in standard position with $p(v_1) = (0, 0)$ and $p(v_2) = (0, y_2)$. Let $K = \mathbb{Q}(d_G(p))$ and $K_1 = K(y_2)$. Since y_2 satisfies the quadratic equation $y_2^2 -$

$d_p(v_1, v_2) = 0$, and since $d_p(v_1, v_2) \in K$ by Lemma 6.1, we have $[K_1 : K] \leq 2$. Let $p(v_i) = (x_i, y_i)$ for all $3 \leq i \leq n$. Then $x_i^2 + y_i^2 = d_p(v_i, v_0)$ and $x_i^2 + (y_i - y_2)^2 = d_p(v_i, v_1)$. Since G is globally rigid, v_i is globally linked to both v_1 and v_2 in G and hence, by Lemma 6.1, $\{d_p(v_i, v_0), d_p(v_i, v_1)\} \subset K$. This implies that $y_i \in K_1$ and $x_i^2 \in K_1$. Since this holds for all $3 \leq i \leq n$, (G, p) is quadratically solvable. \bullet

7 3-connected graphs

A graph $G = (V, E)$ is k -connected if $|V| \geq k + 1$ and $G - U$ is connected for all $U \subseteq V$ with $|U| < k$. We conjecture that a 3-connected graph is radically (or quadratically) solvable if and only if it is globally rigid. We will verify this conjecture for planar graphs. In addition we show that our conjecture is equivalent to an old conjecture of the second author (that no 3-connected minimally rigid graph is radically solvable). We will use the following lemma which tells us that the radical, respectively quadratic, solvability of a rigid graph is preserved by the operation of replacing a subgraph by a radically, respectively quadratically, solvable rigid subgraph. (In our application the new subgraph will be minimally rigid.)

Lemma 7.1 *Let H_0, H_1, H_2 be graphs with $V(H_0) \cap V(H_1) = V(H_0) \cap V(H_2) = V(H_1) \cap V(H_2) = U$, $|U| \geq 2$, and $E(H_0) \cap E(H_1) = E(H_0) \cap E(H_2) = \emptyset$. Let $G_1 = H_0 \cup H_1$ and $G_2 = H_0 \cup H_2$. Suppose that G_1 and H_2 are both rigid. Then*

- (a) G_2 is rigid.
- (b) If G_1 and H_2 are both radically, respectively quadratically, solvable then G_2 is radically, respectively quadratically, solvable.

Proof. Choose $v_1, v_2 \in U$ and let $(G_1 \cup G_2, p)$ be a quasi-generic *real* realisation of $G_1 \cup G_2$ with $p(v_1) = (0, 0)$ and $p(v_2) = (0, y)$ for some $y \in \mathbb{R}$. Let $V_i = V(H_i) \setminus U$ for $0 \leq i \leq 2$.

Suppose that G_2 is not rigid. Since H_2 is rigid, there exists a non-zero infinitesimal motion z_2 of (G_2, p) in \mathbb{R}^2 which keeps H_2 fixed. Then $z_1 : V(G_1) \rightarrow \mathbb{R}^2$ by $z_1(v) = (0, 0)$ for $v \in V(H_1)$ and $z_1(v) = z_2(v)$ for $v \in V(H_0)$ is a non-zero infinitesimal motion of G_1 which keeps H_1 fixed. This contradicts the hypothesis that G_1 is rigid and completes the proof of (a).

Suppose that G_1 and H_2 are both radically, respectively quadratically, solvable. The first assumption implies that $\mathbb{Q}(p|_{V_0}, p|_U, p|_{V_1})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_0}(p), d_{H_1}(p))$. Since the components of $(p|_{V_0}, y, p|_{U \setminus \{v_1, v_2\}}, p|_{V_1})$ are algebraically independent over \mathbb{Q} we may treat them as if they were indeterminates and apply Lemma 3.6 with $X = p|_{V_0}$, $Y = (y, p|_{U \setminus \{v_1, v_2\}})$, $Z = p|_{V_1}$, $f = d_{H_0}(p)$, and $g = d_{H_1}(p)$ to deduce that $\mathbb{Q}(p|_{V_0}, p|_U)$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_0}(p), p|_U)$. We also have $\mathbb{Q}(p|_U, p|_{V_2})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_2}(p))$ by the second assumption. Hence $\mathbb{Q}(d_{H_0}(p), p|_U, p|_{V_2})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_0}(p), d_{H_2}(p))$. Since the components of $(p|_{V_0}, y, p|_{U \setminus \{v_1, v_2\}}, p|_{V_2})$ are algebraically independent over \mathbb{Q} , we may apply Lemma 3.6, with $X = p|_{V_0}$, $Y = (y, p|_{U \setminus \{v_1, v_2\}})$, $Z = p|_{V_2}$, $f = d_{H_0}(p)$, and $g = d_{H_2}(p)$, to deduce that $\mathbb{Q}(p|_{V_0}, p|_U, p|_{V_2})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_0}(p), d_{H_2}(p))$. Thus G_2 is radically, respectively quadratically, solvable and (b) holds. \bullet

We also need a result on graph connectivity due to W. Mader.

Lemma 7.2 [7, Satz 1] *Let G be a k -connected graph and C be a cycle in G such that each vertex of C has degree at least $k + 1$ in G . Then $G - e$ is k -connected for some $e \in E(C)$.*

For $n \geq 4$, the *wheel on n vertices* is the graph $W = (V, E)$ with $V = \{v, u_1, \dots, u_{n-1}\}$ and $E = \{vu_1, vu_2, \dots, vu_{n-1}\} \cup \{u_1u_2, u_2u_3, \dots, u_{n-1}u_1\}$. We refer to the cycle $C = u_1u_2 \dots u_{n-1}u_1$ as the *rim* of W , and to the vertices of C as the *rim vertices* of W .

Lemma 7.3 *Let H_0, H_1 be graphs with $V(H_0) \cap V(H_1) = U$, $|U| \geq 3$, and $E(H_0) \cap E(H_1) = \emptyset$. Let H_2 be a wheel with U as its set of rim vertices, $V(H_0) \cap V(H_2) = U$ and $E(H_0) \cap E(H_2) = \emptyset$. Put $G_1 = H_0 \cup H_1$ and $G_2 = H_0 \cup H_2$. Suppose that G_1 is 3-connected and that each vertex of U has degree at least four in G_2 . Then $G_2 - e$ is 3-connected for some edge e on the rim of H_2 . Furthermore, if G_1 is planar and H_1 is connected, then we may choose H_2 in such a way such that $G_2 - e$ is planar and 3-connected.*

Proof. We first show that G_2 is 3-connected. Suppose not. Then $G_2 - T$ is disconnected for some $T \subseteq V(G_2)$ with $|T| \leq 2$. Since H_2 is 3-connected,

$H_2 - T$ is connected. Hence $H_2 - T$ is contained in a single connected component of $G_2 - T$. This implies that $G_1 - (T \cap V(G_1))$ is disconnected and contradicts the hypothesis that G_1 is 3-connected.

We may now use Lemma 7.2 and the hypothesis that each vertex of U has degree at least four in G_2 to deduce that $G_2 - e$ is 3-connected for some edge e of C .

Finally, we suppose that G_1 is planar and H_1 is connected. Then the vertices of U must lie on the same face F of $G - (V(H_1) - U)$. If we choose H_2 such that, in the above definition of a wheel, the rim vertices u_1, u_2, \dots, u_{n-1} occur in this order around F , then the resulting G_2 will be planar. •

Lemma 7.4 *Let G be obtained by deleting an edge from the rim of a wheel on $n \geq 4$ vertices. Then G is both minimally rigid and quadratically solvable.*

Proof. It is easy to check that G can be obtained from K_3 by recursively adding vertices of degree two. The lemma now follows since K_3 is minimally rigid and quadratically solvable, and the operation of adding a vertex of degree two is known to preserve the properties of being minimally rigid, see [15], and quadratically solvable [9]. •

A graph $G = (V, E)$ is *redundantly rigid* if $G - e$ is rigid for all $e \in E$. A *non-trivial redundantly rigid component* of G is a maximal redundantly rigid subgraph of G . Edges e of G such that $G - e$ is not rigid belong to no redundantly rigid subgraphs of G . We consider the subgraph consisting of such an edge e and its end-vertices to be a *trivial redundantly rigid component*. Thus G is minimally rigid if and only if all its redundantly rigid components are trivial and, when $|V| \geq 3$, G is redundantly rigid if and only if it has exactly one redundantly rigid component.

We can now characterise quadratic solvability in 3-connected planar graphs. We use the fact that a rigid graph $G = (V, E)$ is minimally rigid if and only if $|E| = 2|V| - 3$, see [15].

Theorem 7.5 *Let $G = (V, E)$ be a rigid 3-connected planar graph. Then the following statements are equivalent.*

- (a) G is quadratically solvable.
- (b) G is radically solvable.
- (c) G is redundantly rigid.
- (d) G is globally rigid.

Proof. If G is redundantly rigid then G is globally rigid by [4] and hence is quadratically solvable by Theorem 6.2. Hence (c) implies (d) and (d) implies (a). Clearly (a) implies (b). It remains to show that (b) implies (c). We will prove the contrapositive.

Suppose that G is not redundantly rigid. We show by induction on $|E| - 2|V| + 3$ that G is not quadratically solvable. Since G is rigid we have $|E| - 2|V| + 3 \geq 0$. If equality holds then G is minimally rigid and [10] implies that G is not radically solvable. Hence we may suppose that $|E| > 2|V| - 3$. Then some redundantly rigid component $H_1 = (V_1, E_1)$ of G is non-trivial. Let U be the set of vertices of H_1 which are incident to edges of $E \setminus E_1$ and put $H_0 = (G - E_1) - (V_1 \setminus U)$. By Lemma 7.3, we can choose a wheel W with rim vertices U and an edge e on the rim of W such that $G' = H_0 \cup (W - e)$ is 3-connected and planar. Lemmas 7.1(a) and 7.4 imply that G' is rigid. Since G' is not redundantly rigid and $|V(G')| - 2|E(G')| + 3 < |E| - 2|V| + 3$, we may apply induction to deduce that G' is not radically solvable. Lemmas 7.1(b) and 7.4 now imply that G is not radically solvable. •

We conjecture that the planarity condition can be removed from Theorem 7.5.

Conjecture 7.6 *Let $G = (V, E)$ be a rigid 3-connected graph. Then the following statements are equivalent.*

- (a) G is quadratically solvable.
- (b) G is radically solvable.
- (c) G is redundantly rigid.
- (d) G is globally rigid.

We may use the proof technique of Theorem 7.5 to reduce this conjecture to the special case when G is minimally rigid. This special case was suggested over twenty years ago by the second author.

Conjecture 7.7 [9] *No 3-connected minimally rigid graph is radically solvable.*

We have verified that the smallest 3-connected non-planar minimally rigid graph, $K_{3,3}$, is not radically solvable using a similar proof technique to that used for the prism, or doublet, graph in [10, Theorem 8.4].

8 2-connected graphs

Every rigid graph is 2-connected but not necessarily 3-connected. We show in this section that the problem of deciding whether a minimally rigid graph is radically, respectively quadratically, solvable can be reduced to the special case of 3-connected minimally rigid graphs. We obtain similar reduction results for arbitrary rigid graphs but in this case the reduction to 3-connected graphs is not complete.

Theorem 8.1 *Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be graphs with $V_1 \cap V_2 = \{v_1, v_2\}$ and $E_1 \cap E_2 = \emptyset$. Let $G = H_1 \cup H_2$ and suppose that G is rigid.*

(a) Suppose that H_1, H_2 are both rigid. Then G is radically, respectively quadratically, solvable if and only if $H_1 + v_1v_2, H_2 + v_1v_2$ are both radically, respectively quadratically, solvable.

(b) Suppose that H_1 is not rigid. Then $H_1 + v_1v_2$ and H_2 are both rigid. Furthermore:

(i) if $H_1 + v_1v_2$ and H_2 are both radically, respectively quadratically, solvable then G is radically, respectively quadratically, solvable;

(ii) if G is radically, respectively quadratically, solvable then $H_1 + v_1v_2$ and $H_2 + v_1v_2$ are both radically, respectively quadratically, solvable.

(iii) if G is radically, respectively quadratically, solvable and $H_1 + v_1v_2$ is minimally rigid, then $H_1 + v_1v_2$ and H_2 are both radically, respectively quadratically, solvable.

Proof. Choose a quasi-generic realisation (G, p) of G with $p(v_1) = (0, 0)$ and $p(v_2) = (0, y)$ for some $y \in \mathbb{C}$.

(a) Suppose that $H_1 + v_1v_2$ and $H_2 + v_1v_2$ are both radically, respectively quadratically, solvable. Then $\mathbb{Q}(p|_{V_i})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_i+v_1v_2}(p))$ for $i = 1, 2$. It follows that $\mathbb{Q}(p)$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{G+v_1v_2}(p))$. Since H_1, H_2 are both rigid, [5, Lemma 8.2] implies that v_1 and v_2 are globally linked in (G, p) . By Lemma 6.1, $d_p(v_1, v_2) \in \mathbb{Q}(d_G(p))$ and hence $\mathbb{Q}(d_{G+v_1v_2}(p)) = \mathbb{Q}(d_G(p))$. Thus $\mathbb{Q}(p)$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_G(p))$ and G is radically, respectively quadratically, solvable.

Suppose on the other hand that G is radically, respectively quadratically, solvable. Then $\mathbb{Q}(p)$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_G(p))$. Since (G, p) is quasi-generic, we may apply Lemma 3.6

with $f = d_{H_1}(p)$, $g = d_{H_2}(p)$, $X = p|_{V_1 \setminus \{v_1, v_2\}}$, $Y = y$, and $Z = p|_{V_2 \setminus \{v_1, v_2\}}$ to deduce that $\mathbb{Q}(p|_{V_1})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_1}(p), y)$. Thus $\mathbb{Q}(p|_{V_1})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_1+v_1v_2}(p))$ and so $H_1 + v_1v_2$ is radically, respectively quadratically, solvable. By symmetry, $H_2 + v_1v_2$ is also radically, respectively quadratically, solvable.

(b) The fact that $H_1 + v_1v_2$ and H_2 are both rigid follows from [5, Lemma 8.5].

Suppose that $H_1 + v_1v_2$ and H_2 are both radically, respectively quadratically, solvable. Then $\mathbb{Q}(p|_{V_2})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_2}(p))$. We also have $\mathbb{Q}(p|_{V_1})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_1+v_1v_2}(p))$. Since $y \in \mathbb{Q}(p|_{V_2})$, we have $\mathbb{Q}(p|_{V_1}, p|_{V_2})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_1}(p), p|_{V_2})$. Thus $\mathbb{Q}(p)$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_G(p))$ and G is radically, respectively quadratically, solvable. Hence (i) holds.

Suppose on the other hand that that G is radically, respectively quadratically, solvable. Then $\mathbb{Q}(p)$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_G(p))$. We may apply the argument used in the second part of the proof of (a) to deduce that $\mathbb{Q}(p|_{V_1})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_1+v_1v_2}(p))$, and $\mathbb{Q}(p|_{V_2})$ is a radically, respectively quadratically, solvable extension of $\mathbb{Q}(d_{H_2+v_1v_2}(p))$. Hence (ii) holds.

To prove (iii) we need to show that y belongs to a radical, respectively quadratic, extension of $\mathbb{Q}(d_{H_2}(p))$ when $H_1 + v_1v_2$ is minimally rigid. In this case [5, Lemma 5.6] implies that $X = d_{H_1}(p)$ is algebraically independent over $\mathbb{Q}(d_{H_2+v_1v_2}(p))$. Let $K = \mathbb{Q}(d_{H_2}(p))$ and $L = K(y)$. Since G is radically, respectively quadratically, solvable, $L(X) : K(X)$ is radically, respectively quadratically, solvable. Since X is algebraically independent over L , Lemma 3.5 implies that $L : K$ is radically, respectively quadratically, solvable. Part (iii) now follows since $y \in L$. •

We do not know whether the hypothesis that $H_1 + v_1v_2$ is minimally rigid can be removed from Theorem 8.1(b)(iii). The difficulty in extending the above proof when $H_1 + v_1v_2$ is not minimally rigid is that $d_{H_1}(p)$ will not be algebraically independent over $\mathbb{Q}(d_{H_2+v_1v_2}(p))$. So it is conceivable that $\mathbb{Q}(d_{H_1}(p))$ may contain algebraic numbers which enable y to belong to

a radical extension of $\mathbb{Q}(d_G(p))$ but not to a radical extension of $\mathbb{Q}(d_{H_2}(p))$. On the other hand, we will see in the next section that we can side step this problem and still obtain a characterization of radically solvable rigid graphs if Conjecture 6.6 is true. We will accomplish this by only considering certain separations (H_1, H_2) of G and applying the following result.

Corollary 8.2 *Let $H_i = (V_i, E_i)$ be graphs with $V_i \cap V_j = \{v_k\}$ and $V_1 \cap V_2 \cap V_3 = \emptyset = E_i \cap E_j$ for all $\{i, j, k\} = \{1, 2, 3\}$. Let $G = H_1 \cup H_2 \cup H_3$ and suppose that G is rigid. Then H_1, H_2, H_3 are rigid. Furthermore, G is radically, respectively quadratically, solvable if and only if H_1, H_2, H_3 are radically, respectively quadratically, solvable.*

Proof. Since $G = (H_1 \cup H_2) \cup H_3$ is rigid and $H_1 \cup H_2$ is not rigid, Theorem 8.1(b) implies that H_3 is rigid. We may now use symmetry to deduce that H_1, H_2 are also rigid.

Suppose G is radically, respectively quadratically, solvable. By Theorem 8.1(b)(ii), $(H_1 \cup H_2) + v_1v_2$ is radically, respectively quadratically, solvable. Since $(H_1 \cup H_2) + v_1v_2 = H_1 \cup (H_2 + v_2 + v_1v_2)$ we may again use Theorem 8.1(b)(ii) to deduce that $H_2 + v_2 + v_2v_3 + v_1v_2$ is radically, respectively quadratically, solvable. We can now express $H_2 + v_2 + v_2v_3 + v_1v_2$ as $(K_3 - v_1v_3) \cup H_2$ where $V(K_3) = \{v_1, v_2, v_3\}$. Since K_3 is minimally rigid, we may apply Theorem 8.1(b)(iii) to deduce that H_2 is radically, respectively quadratically, solvable. By symmetry H_1, H_3 are also radically, respectively quadratically, solvable.

Suppose on the other hand that H_1, H_2, H_3 are radically, respectively quadratically, solvable. Let K_3 be a complete graph with $V(K_3) = \{v_1, v_2, v_3\}$. Then K_3 is quadratically solvable, so by Theorem 8.1(b)(i), $F_1 = (K_3 - v_1v_3) \cup H_2$ is radically, respectively quadratically, solvable. We may now apply Theorem 8.1(b)(i) to $F_2 = (F_1 - v_2v_3) \cup H_1$ to deduce that F_2 is radically, respectively quadratically, solvable. Finally we apply Theorem 8.1(b)(i) to $G = (F_2 - v_1v_2) \cup H_3$ to deduce that G is radically, respectively quadratically, solvable. •

9 A family of quadratically solvable graphs

We can recursively construct a family \mathcal{F} of quadratically solvable graphs as follows. We first put all globally rigid graphs in \mathcal{F} . Then, for any $G_1 =$

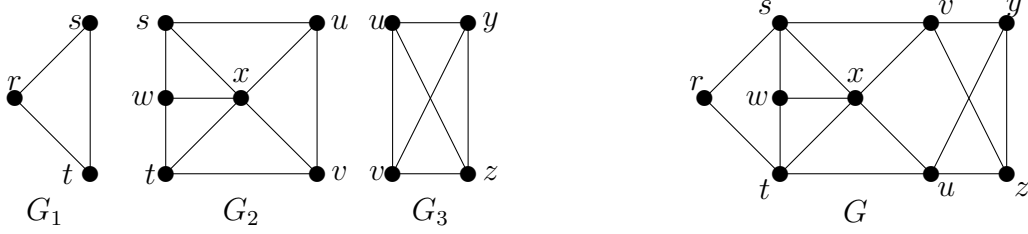


Figure 2: Three globally rigid graphs G_1, G_2, G_3 are combined to give a graph G in \mathcal{F} . We first construct $G_4 = (G_1 - st) \cup G_2$ using operation (b). We then construct $G = (G_4 - uv) \cup (G_3 - uv)$ using operation (c).

(V_1, E_1) and $G_2 = (V_2, E_2)$ in \mathcal{F} with $V_1 \cap V_2 = \{u, v\}$ and $|V_1|, |V_2| \geq 3$ we put:

- (a) $G_1 \cup G_2$ in \mathcal{F} ;
- (b) $(G_1 - e) \cup G_2$ in \mathcal{F} if $e = uv \in E_1$;
- (c) $(G_1 - e) \cup (G_2 - e)$ in \mathcal{F} if $e = uv \in E_1 \cap E_2$ and $G_1 - e, G_2 - e$ are both rigid.

This construction is illustrated in Figure 2. (Note that a recursive construction for globally rigid graphs is given in [4].)

Lemma 9.1 *Every graph in \mathcal{F} is rigid and quadratically solvable.*

Proof. Suppose $G \in \mathcal{F}$. We show that G is rigid and quadratically solvable by induction on $|E|$. If G is globally rigid then G is rigid, and is quadratically solvable by Theorem 6.2. Hence we may suppose that G is not globally rigid. The definition of \mathcal{F} now implies that there exist graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in \mathcal{F} with $V_1 \cap V_2 = \{u, v\}$ and either $G = G_1 \cup G_2$, or $e = uv \in E_1$ and $G = (G_1 - e) \cup G_2$, or $e = uv \in E_1 \cap E_2$, $G_1 - e, G_2 - e$ are both rigid and $G = (G_1 - e) \cup (G_2 - e)$. By induction G_1 and G_2 are both rigid and quadratically solvable.

We first show that G is rigid. Since G_1, G_2 are rigid and $|V_1 \cap V_2| \geq 2$, $G_1 \cup G_2$ is rigid. Furthermore, if $e = uv \in E_1$ then e is a redundant edge in $G_1 \cup G_2$, so $(G_1 - e) \cup G_2$ is also rigid. Finally, if $e \in E_1 \cap E_2$ and $G_1 - e$ and $G_2 - e$ are both rigid then $(G_1 - e) \cup (G_2 - e)$ is rigid. Hence G is rigid.

It remains to show that G is quadratically solvable. Since G_1 and G_2 are quadratically solvable, $G_1 + uv$ and $G_2 + uv$ are quadratically solvable. Hence $G_1 \cup G_2$ is quadratically solvable by Theorem 8.1(a). Suppose that $e = uv \in E_1$ and let $H_1 = G_1 - e$ and $H_2 = G_2$. We can deduce that $G = H_1 \cup H_2$ is quadratically solvable by applying Theorem 8.1(a) to H_1 and $H_2 + uv$ if H_1 is rigid, and by applying Theorem 8.1(b)(i) to H_1 and H_2 if H_1 is not rigid. Thus $(G_1 - e) \cup G_2$ is quadratically solvable. Finally we suppose that $e \in E_1 \cap E_2$ and $G_1 - e, G_2 - e$ are both rigid. Then $(G_1 - e) \cup (G_2 - e)$ is quadratically solvable by Theorem 8.1(a). \bullet

We can use Theorems 7.5 and 8.1 and Lemma 9.1 to characterize when a rigid planar graph is quadratically solvable. We first need to describe a technique for decomposing a rigid graph into ‘3-connected rigid pieces’. This is a special case of a more general theory of Tutte [14] which decomposes 2-connected graphs into ‘3-connected pieces’.

Every 2-connected graph G which is distinct from K_3 and is not 3-connected has a pair of edge-disjoint subgraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ such that $H_1 \cup H_2 = G$, $|V_1 \cap V_2| = 2$, and $V_1 \setminus V_2 \neq \emptyset \neq V_2 \setminus V_1$. We refer to such a pair of subgraphs (H_1, H_2) as a *2-separation* of G and to the vertex set $V_1 \cap V_2$ as a *2-separator* of G .

Given a rigid graph G with at least three vertices, we recursively construct the set \mathcal{C}_G of *cleavage units* of G as follows. If G is 3-connected or $G = K_3$ then we put $\mathcal{C}_G = \{G\}$. Otherwise G has a 2-separation (H_1, H_2) , where $V(H_1) \cap V(H_2) = \{u, v\}$. In this case $G_1 = H_1 + uv$ and $G_2 = H_2 + uv$ are both rigid by Theorem 8.1(b), and we put $\mathcal{C}_G = \mathcal{C}_{G_1} \cup \mathcal{C}_{G_2}$.⁴ Note that the cleavage units of G may not be subgraphs of G since G_1 and G_2 may not be subgraphs of G . (We have $uv \in E(G_1) \cap E(G_2)$ but we may not have $uv \in E(G)$. For example the cleavage units of the graph G in Figure 2 are G_1 , $G_2 + st$ and G_3 , and none of these are subgraphs of G .)

Lemma 9.2 *Let G be a rigid graph on at least three vertices. Then every cleavage unit of G is either equal to K_3 or is 3-connected and rigid. Furthermore, if G is radically, respectively quadratically, solvable, then every*

⁴In order to obtain a unique decomposition of a 2-connected graph G into cleavage units Tutte [14] only considers *excisable 2-separations* i.e. 2-separations (H_1, H_2) such that at least one of H_1, H_2 is 2-connected. When G is rigid, Theorem 8.1(b) tells us that for every 2-separation (H_1, H_2) , at least one of H_1, H_2 will be rigid (and hence 2-connected) so all 2-separations of a rigid graph are excisable.

cleavage unit of G is radically, respectively quadratically, solvable.

Proof. If G itself is K_3 or is 3-connected then the lemma is trivially true. Hence we may suppose that G has a 2-separation (H_1, H_2) , where $V(H_1) \cap V(H_2) = \{u, v\}$. Theorem 8.1 implies that $H_1 + uv, H_2 + uv$ are both rigid, and are radically, respectively quadratically, solvable if G is radically, respectively quadratically, solvable. The lemma now follows by induction on $|V(G)|$ using the fact that $\mathcal{C}_G = \mathcal{C}_{H_1+uv} \cup \mathcal{C}_{H_2+uv}$. •

We can now obtain our promised characterization of quadratic solvability for rigid planar graphs.

Theorem 9.3 *Let G be a rigid planar graph. Then the following statements are equivalent.*

- (a) *G is quadratically solvable.*
- (b) *G is radically solvable.*
- (c) *G belongs to \mathcal{F} .*

Proof. We have (c) implies (a) by Lemma 9.1, and (a) implies (b) by definition. It remains to show that (b) implies (c). We proceed by contradiction. Suppose there exists a radically solvable rigid planar graph G such that $G \notin \mathcal{F}$. We may assume that G is chosen to have as few vertices as possible (and hence every radically solvable rigid planar graph with fewer vertices than G belongs to \mathcal{F}). Since $G \notin \mathcal{F}$, $G \neq K_2, K_3$. If G were 3-connected then G would be globally rigid by Theorem 7.5 and hence we would have $G \in \mathcal{F}$. Thus G is not 3-connected and we may choose a 2-separation (H_1, H_2) of G , where $V(H_1) \cap V(H_2) = \{u, v\}$. By Theorem 8.1, $H_1 + uv, H_2 + uv$ are both rigid and radically solvable. Since they are also planar and have fewer vertices than G we have $H_1 + uv, H_2 + uv \in \mathcal{F}$. If $uv \in E(G)$ then $G = (H_1 + uv) \cup (H_2 + uv) \in \mathcal{F}$ by operation (a) in the definition of \mathcal{F} . Hence $uv \notin E(G)$. If H_1, H_2 are both rigid then $G = H_1 \cup H_2 \in \mathcal{F}$ by operation (c) in the definition of \mathcal{F} . Thus, for every 2-separator $\{u, v\}$ of G , $uv \notin E(G)$, and for every 2-separation (H_1, H_2) of G , one of H_1 and H_2 is not rigid.

We now modify our choice of the 2-separation (H_1, H_2) if necessary so that H_1 is not rigid and, subject to this condition, H_1 has as few vertices as possible.

Claim 1 *There exists a unique cleavage unit G_1 of G with $\{u, v\} \subset V(G_1) \subseteq V(H_1)$. In addition we have $G_1 = K_3$.*

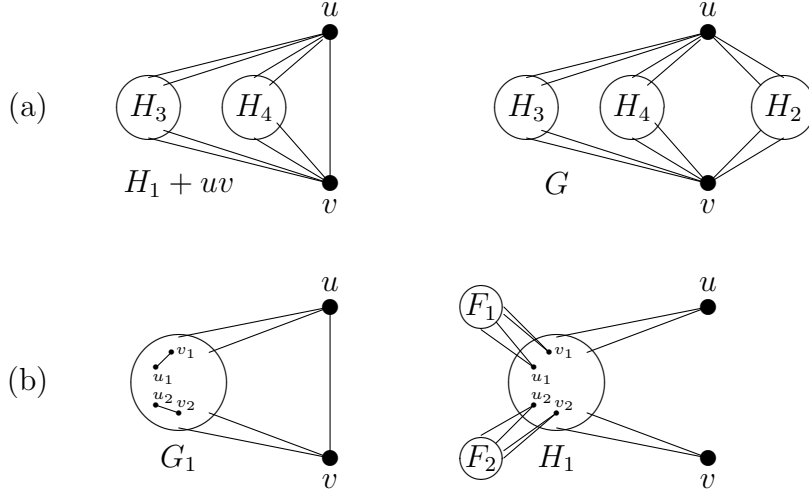


Figure 3: Proof of Claim 1: (a) the case when there are two distinct cleavage units G_3, G_4 of G with $\{u, v\} \subset V(G_i) \subseteq V(H_1)$ for $i = 3, 4$; (b) the case when $G_1 \neq K_3$.

Proof. Suppose that there are two distinct cleavage units G_3, G_4 of G with $\{u, v\} \subset V(G_i) \subseteq V(H_1)$ for $i = 3, 4$. Then $H_1 + uv$ has a 2-separation (H_3, H_4) with $uv \in E(H_4)$, $V(H_3) \cap V(H_4) = \{u, v\}$ and $V(G_i) \subseteq V(H_i)$ for $i = 3, 4$, see Figure 3(a). Since $H_1 = H_3 \cup (H_4 - uv)$ is not rigid, H_3 is not rigid. Thus $(H_3, H_2 \cup (H_4 - uv))$ is a 2-separation of G in which H_3 is not rigid and has fewer vertices than H_1 . This contradicts the choice of (H_1, H_2) . Hence there is a unique cleavage unit G_1 of G with $\{u, v\} \subset V(G_1) \subseteq V(H_1)$.

Suppose $G_1 \neq K_3$. Then G_1 is 3-connected and radically solvable by Lemma 9.2. Since G_1 is planar, Theorem 7.5 now implies that G_1 is redundantly rigid and hence that $G_1 - uv$ is rigid. Let $\{u_i, v_i\}$, $1 \leq i \leq m$, be the 2-separators of $H_1 + uv$ with $\{u_i, v_i\} \subset V(G_1)$. Then $u_i v_i \in E(G_1)$ for $1 \leq i \leq m$, see Figure 3(b). For each $1 \leq i \leq m$ we may choose a 2-separation (F_i, F'_i) of $H_1 + uv$ with $V(G_1) \subset V(F'_i)$. Then $(F_i, (F'_i - uv) \cup H_2)$ is a 2-separation of G . The choice of H_1 and the fact that F_i is properly contained in H_1 now implies that F_i is rigid for all $1 \leq i \leq m$. Since $G_1 - uv$ is rigid, this implies that

$$H_1 = [(G_1 - uv) - \{u_i v_i : 1 \leq i \leq m\}] \cup \bigcup_{i=1}^m F_i$$

is rigid. This contradicts the choice of H_1 . Thus $G_1 = K_3$. •

We can now complete the proof of the theorem. Since $G_1 = K_3$ we can express G as $G = H'_1 \cup H''_1 \cup H_2$ where $H'_1 \cup H''_1 = H_1$, $V(H'_1) \cap V(H_2) = \{u\}$, $V(H''_1) \cap V(H_2) = \{v\}$, $V(H'_1) \cap V(H''_1) = \{w\}$ for some $w \in V(H_1) \setminus \{u, v\}$, and H'_1, H''_1, H_2 are pairwise edge-disjoint. Corollary 8.2 now implies that H'_1, H''_1, H_2 are rigid and radically solvable. Since they are planar and have fewer vertices than G , we have $H'_1, H''_1, H_2 \in \mathcal{F}$. Since G can be obtained from K_3, H'_1, H''_1, H_2 by applying operation (b) in the definition of \mathcal{F} at most three times, we have $G \in \mathcal{F}$. This contradicts the choice of G . •

Since the operations (a), (b) and (c) used in the construction of \mathcal{F} preserve planarity, Theorem 9.3 implies that the family of quadratically solvable planar graphs can be constructed recursively from the family of globally rigid planar graphs by applying operations (a), (b) and (c).

We conjecture that the planarity hypothesis can be removed from Theorem 9.3.

Conjecture 9.4 *Let G be a rigid graph. Then the following statements are equivalent.*

- (a) *G is quadratically solvable.*
- (b) *G is radically solvable.*
- (c) *G belongs to \mathcal{F} .*

The proof technique of Theorem 9.3 can be used to show that Conjecture 9.4 is equivalent to Conjecture 7.6, and hence is also equivalent to Conjecture 7.7.

The constructions and some of the results of this section extend earlier work of the second author for minimally rigid graphs which is implicitly given in [9], and explicitly stated in [10, Theorem 3.2]. He recursively constructs a subfamily, \mathcal{F}_{iso} , of \mathcal{F} as follows. He first puts K_3 in \mathcal{F}_{iso} . Then, for any $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in \mathcal{F}_{iso} with $V_1 \cap V_2 = \{u, v\}$ and $e = uv \in E_1$ he puts $(G_1 - e) \cup G_2$ in \mathcal{F}_{iso} . He shows that every graph in \mathcal{F}_{iso} is minimally rigid and quadratically solvable and conjectures that every radically solvable minimally rigid graph belongs to \mathcal{F}_{iso} .⁵

⁵Since the radical solvability of a graph is preserved by the addition of edges, it is tempting to also conjecture that a graph is radically solvable if and only if it has a spanning

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subgraph in \mathcal{F}_{iso} . This is not the case however. The complete bipartite graph $K_{3,4}$ is globally rigid and hence quadratically solvable, but for each edge e , $K_{3,4} - e$ is minimally rigid and does not belong to \mathcal{F}_{iso} . In addition, we can use Theorem 8.1(b)(iii) and the fact that $K_{3,3}$ is not radically solvable to deduce that $K_{3,4} - e$ is not radically solvable, so $K_{3,4}$ is ‘minimally radically solvable’ but not minimally rigid.

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